

A FRAMEWORK FOR TROPICAL MIRROR SYMMETRY

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ABSTRACT. Applying tropical geometry a framework for mirror symmetry including a mirror construction for Calabi-Yau varieties was proposed by the author. We discuss the conceptual foundations of this construction based on a natural mirror map identifying deformations and divisors. We show how the construction specializes to that by Batyrev for hypersurfaces and its generalization by Batyrev and Borisov to complete intersections. Based on an explicit example we comment on the implementation in the MACAULAY2 package SRDEFORMATIONS.

1. INTRODUCTION

Mirror symmetry is a key link between mathematics and theoretical physics, e.g., algebraic geometry obtains new ideas in enumerative geometry from superstring theory which, in return, benefits from the study of Calabi-Yau varieties. Important insight to mirror symmetry is gained by explicit constructions computing for a given Calabi-Yau variety the corresponding mirror Calabi-Yau, for a general account of the topic see [16].

The mirror of the general quintic hypersurface in \mathbb{P}^4 was given by Greene and Plesser [19] as an orbifold of a 1-parameter family of quintics. For toric hypersurfaces this class of orbifolding constructions was unified by Batyrev [4] using the involution of Gorenstein toric Fano varieties given by dualization of reflexive polyhedra. Batyrev's description proved to be well suited for the study of further properties like mirror duality of stringy Hodge numbers [5], Picard-Fuchs equations [10] and much more. It was generalized by Batyrev and Borisov to complete intersections [6, 7] using nef partitions and by Batyrev and Nill [8] via reflexive Gorenstein cones.

Based on ideas of Leung and Vafa [27] and Kontsevich and Soibelman [26], Gross and Siebert [22, 23, 24] used toric degenerations and integrally affine manifolds to give a mirror construction, which is expected to eventually relate B -model period integrals and tropically counted A -model Gromov-Witten invariants. For a first instance of this in the case of \mathbb{P}^2 see [21]. For the fundamental idea of tropical curve counting see Mikhalkin [28]. In [20] Gross shows how to construct complete intersection mirrors. In order to apply the Gross-Siebert program one has to obtain simple affine structures, which is achieved by considering more general families, which become toric degenerations after desingularization. For an independent construction of these integral affine structures in the complete intersection case by Haase see [25].

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In [12] the author developed via embedded tropical varieties a general framework for mirror symmetry leading to an algorithmic mirror construction. It directly specializes to the known constructions by Batyrev for hypersurfaces and its generalization by Batyrev and Borisov to complete intersections, and reproduces that by Rødland [29] for a Pfaffian non-complete intersection. The tropical mirror construction extends this construction to a considerably larger class of Calabi-Yau varieties and produces explicit new mirror examples [12, Sec. 10.5]. It comes with a natural mirror map identifying deformations and divisors. In this paper we focus on the conceptual foundation of the tropical mirror construction and how to recover the Batyrev-Borisov mirror of a complete intersection. We also discuss the implementation in the MACAULAY2 [18] package SRDEFORMATIONS [13].

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2. INGREDIENTS FROM TORIC AND TROPICAL GEOMETRY, AND DEFORMATION THEORY

In this section we discuss basic facts from toric and tropical geometry and deformation theory necessary to formulate the tropical mirror construction, and fix some notation in this context.

2.1. Toric geometry. We introduce the basic toric objects used in the tropical mirror construction. For more details on toric geometry see, e.g., [16] and [36].

2.1.1. Toric Fano varieties. A **Fano polytope** $P \subset N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ in the lattice $N = \mathbb{Z}^n$ is an integral polytope which contains 0 as its unique interior lattice point. The fan $\Sigma = \Sigma(P)$ over P given by the cones $\text{posHull } F$ spanned by the faces F of P defines a \mathbb{Q} -Gorenstein toric Fano variety $Y = \text{TV}(\Sigma)$ of dimension n . The toric strata of Y correspond to the faces of the dual polytope $\Delta = P^* \subset M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$ where $M = \text{Hom}(N, \mathbb{Z})$.

Denote by $\mathbb{P}(\Delta)$ the projective toric variety defined an integral polytope $\Delta \subset M_{\mathbb{R}}$ and by $\Sigma = \text{NF}(\Delta) = \Sigma(P)$ the normal fan of Δ , so $\mathbb{P}(\Delta) \cong \text{TV}(\Sigma)$ and $\mathcal{O}_{\mathbb{P}(\Delta)}(1) \cong \mathcal{O}_{\mathbb{P}(\Delta)}(D_{\Delta})$ with the Cartier divisor

$$D_{\Delta} = \sum_{r \in \Sigma(1)} - \min_{m \in \Delta} \langle m, \hat{r} \rangle D_r$$

Here we denote by $\Sigma(1)$ the set of rays (cones of dimension 1) of Σ , by D_r the torus invariant prime Weil divisor corresponding to $r \in \Sigma(1)$ and by \hat{r} the minimal lattice generator of r .

A polytope $\Delta \subset M_{\mathbb{R}}$ of dimension n is called **reflexive** if Δ and its dual Δ^* are integral and contain 0 in their interior. Then $\mathbb{P}(\Delta)$ and $\mathbb{P}(\Delta^*)$ are Gorenstein toric Fano varieties.

2.1.2. Cox ring. Subvarieties of a toric variety $Y = \text{TV}(\Sigma)$ can be described by ideals in the **Cox ring** (or homogeneous coordinate ring) $S = \mathbb{C}[x_r \mid r \in \Sigma(1)]$ of Y , see [15]. This is a polynomial ring with one variable x_r for each ray $r \in \Sigma(1)$ graded by the presentation sequence

$$(2.1) \quad 0 \rightarrow M \xrightarrow{A} \mathbb{Z}^{\Sigma(1)} \xrightarrow{\deg} A_{n-1}(Y) \rightarrow 0$$

of the Chow group $A_{n-1}(Y)$ of classes $[D]$ of Weil divisors D modulo linear equivalence. The rows of the matrix A are given by the minimal lattice generators of the rays of Σ . In terms of monomials

$$\deg(\prod_r x_r^{a_r}) = [\sum_r a_r D_r]$$

Proposition 1. The vector space of global sections of the reflexive sheaf of sections $\mathcal{O}_Y(D)$ of a Weil divisor D in Y is isomorphic to the degree $[D]$ -part of the Cox ring

$$H^0(Y, \mathcal{O}_Y(D)) \cong S_{[D]}$$

We denote by $\Delta_D \subset M_{\mathbb{R}}$ the polytope of sections of a divisor D , i.e., the convex hull of the torus invariant sections.

From $A_{n-1}(Y)$, depending only on the rays of Σ , and the **irrelevant ideal**

$$B(\Sigma) = \left\langle \prod_{r \in \Sigma(1), r \notin \sigma} x_r \mid \sigma \in \Sigma \right\rangle \subset S,$$

the toric variety Y can be recovered as the categorical quotient

$$Y = \left(\mathbb{C}^{\Sigma(1)} - V(B(\Sigma)) \right) // \text{Hom}_{\mathbb{Z}}(A_{n-1}(Y), \mathbb{C}^*)$$

Definition 2. If $I \subset S$ is generated by homogeneous elements $f \in S$ with $\deg(f) \in \text{Pic}(Y)$, then I is called **Pic(Y)-generated**. The ideal I is called **Pic(Y)-saturated** if $I_{\alpha} = (I : B(\Sigma)^{\infty})_{\alpha}$ for all $\alpha \in \text{Pic}(Y)$.

Definition 3. The **Picard-Cox ring** of Y is

$$R = \bigoplus_{\alpha \in \text{Pic}(Y)} S_{\alpha}$$

By [15], if Y is simplicial, there is a one-to-one correspondence between the Pic(Y)-generated and Pic(Y)-saturated ideals $I \subset S$ and the closed subschemes of Y . Equivalently one can consider graded ideals of the Picard-Cox ring R , which are saturated in $B(\Sigma) \cap R$.

2.2. Tropical geometry. Tropical geometry will be applied in the mirror construction as a tool to explore one parameter degenerations with fibers in a toric variety, as it associates to such a degeneration a combinatorial object. We recall some basic facts, for more details on tropical geometry see, e.g., [33].

2.2.1. Amoebas. Tropical geometry was motivated by the study of the **amoeba** of a subvariety $V \subset (\mathbb{C}^*)^n$ which is defined as the image of V under the map

$$\begin{aligned} \log_t : (\mathbb{C}^*)^n &\rightarrow \mathbb{R}^n \\ (z_1, \dots, z_n) &\mapsto (\log_t |z_1|, \dots, \log_t |z_n|) \end{aligned}$$

for some base t . Note, that considering the fibers of this map relates tropical geometry to the context of torus fibrations in mirror symmetry. The limit of

the amoeba for $t \rightarrow \infty$ in the Hausdorff metric on compacts can be obtained as a non-Archimedean version of the amoeba:

2.2.2. Tropical varieties. Consider the field of Puiseux series $\mathbb{C}\{\{t\}\}$, which is equipped with the valuation

$$\begin{aligned} \text{val} : \mathbb{C}\{\{t\}\} &\rightarrow \mathbb{Q} \cup \{\infty\} \\ \sum_{j \in J} \alpha_j t^j &\mapsto \min J \end{aligned}$$

and with a norm $\|f\| = e^{-\text{val}(f)}$. Extend val and $\|-\|$ to the metric completion K of $\mathbb{C}\{\{t\}\}$ containing those elements $\sum_{j \in J} \alpha_j t^j$, which satisfy the condition that any subset of J has a minimum. So K is a complete algebraically closed non-Archimedean field with surjective valuation $\text{val} : K \rightarrow \mathbb{R} \cup \{\infty\}$.

Let I be an ideal in $K[x_1, \dots, x_n]$. The image of the algebraic variety $V_K(I) \subset (K^*)^n$ defined by I under the non-Archimedean amoeba map

$$\begin{aligned} \text{val} = \log \|-\| : (K^*)^n &\rightarrow \mathbb{R}^n \\ (x_1, \dots, x_n) &\mapsto (\text{val}(x_1), \dots, \text{val}(x_n)) \end{aligned}$$

is called the non-Archimedean amoeba of $V_K(I)$ or **tropical variety** $T(I)$ of I .

For $w \in \mathbb{R}^n$ the **initial form** $\text{in}_w(f)$ of $f \in K[x_1, \dots, x_n]$ is the sum of the terms of maximal weight with respect to w and $\text{weight}(c) = -\text{val}(c)$ for $c \in K$. For any ideal $J \subset K[x_1, \dots, x_n]$ its **initial ideal** is

$$\text{in}_w(J) = \langle \text{in}_w(f) \mid f \in J \rangle$$

The **tropical semiring** is $\mathbb{R} \cup \{\infty\}$ with tropical addition and multiplication

$$\begin{aligned} a \oplus b &= \min(a, b) \\ a \odot b &= a + b \end{aligned}$$

For any polynomial

$$f = \sum_a b_a(t) \cdot x^a \in K[x_1, \dots, x_n]$$

define its **tropicalization** as the piecewise linear function

$$\text{trop}(f) = \bigoplus_a \text{val}(b_a(t)) \odot x^{\odot a}$$

and by $T(\text{trop}(f))$ its corner locus, i.e., the set of $w \in \mathbb{R}^n$ such that the minimum is attained at least twice. Then the fundamental theorem of tropical geometry is:

Theorem 4. [32, Sec. 9.2], [30, Sec. 2], [33] *If $I \subset K[x_1, \dots, x_n]$ is an ideal, then*

$$\begin{aligned} T(I) &= \{w \in \mathbb{R}^n \mid \text{in}_w(I) \text{ contains no monomial}\} \\ &= \bigcap_{f \in I} T(\text{trop}(f)) \end{aligned}$$

Remark 5. The tropical variety $T(I)$ has a structure of a polyhedral cell complex, its dimension is the Krull dimension of $K[x_1, \dots, x_n]/I$ and it is equidimensional if $V_K(I)$ is.

The tropical variety $T(I)$ is a subset of the space of weight vectors $(w_{x_1}, \dots, w_{x_n}) \in \mathbb{R}^n$ on the monomials of $K[x_1, \dots, x_n]$.

2.3. Bergman fan. From the point of view of Gröbner fans, see for example [31], it is more natural to work with a tropical fan: We give a non-Archimedean definition of the Bergman fan (historically defined via an amoeba type limit). Let $I \subset \mathbb{C}[t][x_1, \dots, x_n]$ be an ideal and denote by L the metric completion of $\mathbb{C}\{\{s\}\}$. The image of $V_L(I)$ under

$$(L^*)^{n+1} \rightarrow \mathbb{R}^{n+1} \\ (t, x_1, \dots, x_n) \mapsto (\text{val}(t), \text{val}(x_1), \dots, \text{val}(x_n))$$

is a fan (the tropical variety of I considering t as a variable), which we denote as the **Bergman fan** of I .

For a fan Σ and a hyperplane H in \mathbb{R}^{n+1} define $\Sigma \cap H$ as the polyhedral cell complex consisting of the faces $\sigma \cap H$ for $\sigma \in \Sigma$. With this notation we immediately get:

Proposition 6. For an ideal $I \subset \mathbb{C}[t][x_1, \dots, x_n]$

$$\text{BF}(I) \cap \{w_t = 1\} = T(I)$$

The intersection with the hyperplane $\{w_t = 1\}$ amounts to identification of the parameter s of the Puiseux series solutions and the parameter t of the degeneration.

2.4. Tropical varieties and the Cox ring. Consider an ideal $I \subset K \otimes S$ where S is the Cox ring of a toric variety Y . As seen in Section 2.2.2, the tropical variety of I should be considered as a subset of the space of weight vectors. Hence in the Cox setup $T(I)$ is naturally a subset of the weight space

$$\frac{\text{Hom}_{\mathbb{R}}(\mathbb{R}^{\Sigma(1)}, \mathbb{R})}{\text{Hom}_{\mathbb{R}}(A_{n-1}(Y) \otimes \mathbb{R}, \mathbb{R})} \xrightarrow[\varphi]{\cong} N_{\mathbb{R}}$$

on S , where the isomorphism φ is obtained from the presentation sequence of the Chow group (Equation 2.1).

For example if $Y = \mathbb{P}^n$ the space of weights will be $\mathbb{R}^{n+1}/\mathbb{R}(1, \dots, 1)$.

Remark 7. When discarding the grading, the tropical variety may still contain linear space after dividing by $\text{Hom}_{\mathbb{R}}(A_{n-1}(Y) \otimes \mathbb{R}, \mathbb{R})$. In some settings it makes sense to divide out also this lineality space, e.g., in the context of tropical Grassmannians [30]. In general however, it should be considered as part of the tropical variety, as then the dimensions of the tropical variety and the algebraic subvariety of Y will coincide.

2.5. Deformations of monomial ideals. Let I_0 be a reduced monomial ideal in the Cox ring S of the toric variety Y . As I_0 is generated by finitely many elements and the space of elements of S of a given degree is finite-dimensional, the degree 0 homomorphisms in $\text{Hom}(I_0, S/I_0)$ form a finite-dimensional vector space denoted by $\text{Hom}(I_0, S/I_0)_0$. The big torus $(\mathbb{C}^*)^{\Sigma(1)}$ acts by

$$\begin{aligned} \text{Hom}_{\mathbb{Z}}(\mathbb{Z}^{\Sigma(1)}, \mathbb{C}^*) \times \mathbb{C}[\mathbb{Z}^{\Sigma(1)}] &\rightarrow \mathbb{C}[\mathbb{Z}^{\Sigma(1)}] \\ (\lambda, m) &\mapsto \lambda(m) \cdot m \end{aligned}$$

on $\mathbb{C}[\mathbb{Z}^{\Sigma(1)}]$ and on S . The induced action of the abelian group $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}^{\Sigma(1)}, \mathbb{C}^*)$ on the vector space $\text{Hom}(I_0, S/I_0)_0$ gives a representation

$$\text{Hom}_{\mathbb{Z}}(\mathbb{Z}^{\Sigma(1)}, \mathbb{C}^*) \rightarrow \text{GL}(\text{Hom}(I_0, S/I_0)_0)$$

which decomposes into characters, as any irreducible representation of an abelian group over an algebraically closed field is 1-dimensional. So the vector space $\text{Hom}(I_0, S/I_0)_0$ has a basis of deformations which are characters. Being of degree 0, any such homomorphism corresponds to an element $\alpha \in M \cong \text{image}(A) \subset \mathbb{Z}^{\Sigma(1)}$ (using the notation of the presentation of the Chow group of divisors of Y from Equation 2.1). On the other hand to any $\alpha \in M$ we can associate a homomorphism $\delta_{\alpha} : I_0 \rightarrow S/I_0$ (which may be 0). See also the results in [2].

3. BASIC FORMULATION OF THE TROPICAL MIRROR CONSTRUCTION

3.1. Conceptual foundation. Mirror symmetry is usually considered in the context of Calabi-Yau varieties, i.e., normal projective algebraic variety with at worst Gorenstein canonical singularities, trivial canonical sheaf $K_X = \Omega_X^d \cong \mathcal{O}_X$ and $h^i(X, \mathcal{O}_X) = 0$ for $0 < i < d$. We will also consider this setup, however the tropical mirror construction is not limited to the Calabi-Yau case.

3.1.1. Complex and Kähler moduli. In physics (see also [16]) a superconformal field theory is associated to a tuple (X, ω) of a Calabi-Yau variety X and a complexified Kähler form $\omega = B + iJ$ with $B, J \in H^2(X, \mathbb{R})$ and J a Kähler class. Mirror symmetry postulates the existence of a mirror dual tuple $(X^{\circ}, \omega^{\circ})$ leading to an isomorphic superconformal field theory.

Keeping ω fixed and varying X should translate into X° being fixed and ω° vary, and vice versa, the identification given by the so called mirror map. Hence locally the complex moduli space of X is being identified with the Kähler moduli space of ω° . So the corresponding tangent spaces $H^1(T_X) = H^{d-1,1}(X)$ of the complex moduli space and $H^{1,1}(X^{\circ})$ of the Kähler moduli space are isomorphic.

3.1.2. Mirror symmetry and degenerations. By this argument, mirror symmetry should be considered in a natural way not as a relation on individual Calabi-Yau varieties, but rather on embedded flat families. This idea is already present, e.g., in the representation of the mirror of the general quintic X as a $h^{2,1}(X^{\circ}) = h^{1,1}(X) = 1$ -parameter family degenerating in the union of 5 planes [19]. It was formalized in [22, 23, 24] in the context of toric degenerations. However note, that some degenerations, one would like to apply mirror symmetry to (e.g., some Pfaffian examples [29], [12]) do not fall in this category.

3.1.3. Basic setup. In the approach presented here we will see mirror symmetry as a correspondence of monomial degenerations of Calabi-Yau varieties. So we consider flat families $\mathfrak{X} \subset Y \times \text{Spec } \mathbb{C}[t]$ with Calabi-Yau fibers $X_t \subset Y$ in a \mathbb{Q} -Gorenstein toric Fano variety Y . The degeneration \mathfrak{X} is specified by an ideal $I \subset \mathbb{C}[t] \otimes S$, homogeneous with respect to the variables of the Cox ring S of Y , and X_0 by a monomial ideal $I_0 \subset S$.

The goal is to associate to \mathfrak{X} a mirror degeneration \mathfrak{X}° with fibers in a mirror toric Fano variety Y° . This will be done in a way, that we obtain a natural mirror map relating $H^{d-1,1}(X)$ and $H^{1,1}(X^\circ)$ for generic fibers X of \mathfrak{X} and X° of \mathfrak{X}° . This can be seen as a generalization of the monomial-divisor mirror map introduced in [3] for hypersurfaces in Gorenstein toric Fano varieties.

For the construction we represent the complex moduli space of X via a one parameter family \mathfrak{X} which is general in the following sense: Consider a big torus invariant basis $v_1, \dots, v_p \in \text{Hom}(I_0, S/I_0)_0$ of degree 0 homomorphisms of the tangent space of the component of the Hilbert scheme of X_0 containing \mathfrak{X} . Suppose that the tangent vector $v = \sum_{i=1}^p \lambda_i v_i$ of \mathfrak{X} satisfies $\lambda_i \neq 0 \forall i$.

Remark 8. One could also consider special subfamilies, e.g., with prescribed singularities. Furthermore, it seems possible to formulate a version of the construction in several parameters t_1, \dots, t_p , which could also be handled by tropical geometry like in Section 2.3. This may avoid representing the moduli space by a one parameter family and eventually could increase the scope of the construction.

3.1.4. Basic idea. As discussed in Section 2.5 the elements v_1, \dots, v_p correspond to elements $\alpha_1, \dots, \alpha_p \in M$ of the lattice of monomials of Y . The basic idea of the tropical mirror construction is to consider the convex hull ∇^* of $\alpha_1, \dots, \alpha_p$ and as Y° the toric variety defined by the fan Σ° over the faces of ∇^* . Hence toric divisors of Y° , and the induced divisors on a prospective mirror inside constructed via tropical geometry, will correspond to deformations of X_0 . On the other hand the deformations of the mirror special fiber X_0° should be induced by the toric divisors of Y .

We now give a short outline of the general tropical mirror construction, for more details see [12].

3.2. Input data. We begin by summarizing the input data:

3.2.1. Toric Fano variety. Let $N = \mathbb{Z}^n$ and $\Delta^* \subset N_{\mathbb{R}}$ be a Fano polytope and $Y = X(\Sigma)$, $\Sigma = \text{Fan}(\Delta^*)$, the corresponding toric Fano variety with Cox ring $S = \mathbb{C}[x_r \mid r \in \Sigma(1)]$ graded by

$$0 \rightarrow M \xrightarrow{A} \mathbb{Z}^{\Sigma(1)} \xrightarrow{\deg} A_{n-1}(Y) \rightarrow 0$$

3.2.2. Monomial Calabi-Yau. Let $X_0 \subset Y$ be given by a reduced, $\text{Pic}(Y)$ -generated monomial ideal $I_0 \subset S$ such that the subcomplex $\text{Strata}_\Delta(I_0) \subset \partial\Delta$ of the boundary complex of Δ , consisting of the toric strata of X_0 , is homeomorphic to a sphere.

3.2.3. Degeneration. Let $\mathfrak{X} \subset Y \times \text{Spec } \mathbb{C}[t]$ be a flat family of Calabi-Yau varieties of dimension d with fibers $X_t \subset Y$ and monomial special fiber X_0 . The degeneration \mathfrak{X} is specified by an ideal $I \subset \mathbb{C}[t] \otimes S$, which is homogeneous with respect to the variables of S .

3.3. Construction of the mirror polarization via Gröbner bases. Fix a monomial ordering $>$ on $\mathbb{C}[t] \otimes S$, which is respecting the Cox grading on S and is local in t , and denote by $>_w$ the weight ordering by w refined by $>$.

Definition 9. The **Gröbner cone of special fiber weights** is defined as the closed cone

$$C_{I_0}(I) = \left\{ -(w_t, w_x) \in \mathbb{R} \oplus N_{\mathbb{R}} \mid L_{>_{(w_t, \varphi(w_x))}}(I) = I_0 \right\}$$

with φ as given in Section 2.2.

Under suitable conditions on the genericity of the degeneration and on the smoothness of the base, the intersection of $C_{I_0}(I)$ with the hyperplane of t -weight one is a polytope and its dual a Fano polytope. We restrict to this case.

Definition 10. The **polytope of special fiber weights** is

$$\nabla_{I_0}(I) = C_{I_0}(I) \cap \{w_t = 1\} \subset N_{\mathbb{R}}$$

The Fano polytope $\nabla_{I_0}(I)^* \subset M_{\mathbb{R}}$ defines a toric Fano variety $Y^\circ = X(\Sigma^\circ)$ of the same dimension as Y by the fan $\Sigma^\circ = \text{Fan}(\nabla_{I_0}(I)^*) \subset M_{\mathbb{R}}$ over the faces of $\nabla_{I_0}(I)^*$. Denote by $S^\circ = \mathbb{C}[z_r \mid r \in \Sigma^\circ(1)]$ the Cox ring of Y° , graded by

$$0 \rightarrow N \xrightarrow{A^\circ} \mathbb{Z}^{\Sigma^\circ(1)} \xrightarrow{\deg} A_{n-1}(Y^\circ) \rightarrow 0$$

3.4. Tropical geometry construction of the mirror degeneration.

3.4.1. *Mirror special fiber.* Denote by $\partial C_{I_0}(I)$ the fan of all boundary faces of the cone $C_{I_0}(I)$.

Definition 11. Consider the fan $BF_{I_0}(I) = BF(I) \cap \partial C_{I_0}(I)$ of the tropical faces of $C_{I_0}(I)$. By intersecting all cones of $BF_{I_0}(I)$ with the hyperplane $\{w_t = 1\}$ one obtains a subcomplex $T_{I_0}(I) \subset \partial \nabla_{I_0}(I)$, which we will denote as the **special fiber tropical variety**.

The support of $T_{I_0}(I)$ is a subset of the tropical variety of I . The complex $T_{I_0}(I)$ is a subdivision of the dual sphere of $\text{Strata}_\Delta(I_0)$.

Recall, that there is a one-to-one correspondence between the Cox variables y_r of Y° , the rays $r \in \Sigma^\circ(1)$ and the facets F_r° of ∇ . Associated to the complex $T_{I_0}(I) \subset \partial \nabla$ of dimension d we have a reduced monomial ideal

$$(3.1) \quad \begin{aligned} I_0^\circ &= \left\langle \prod_{r \in J} y_r \mid J \subset \Sigma^\circ(1), \text{supp}(T_{I_0}(I)) \subset \bigcup_{r \in J} F_r^\circ \right\rangle \\ &= \bigcap_{F^\circ \text{ facet of } T_{I_0}(I)} \langle y_{G^*} \mid G \text{ a facet of } \nabla \text{ with } F^\circ \subset G \rangle \end{aligned}$$

defining a monomial Calabi-Yau of equi-dimension d in Y° . The ideal is Σ -saturated, i.e., all primary components are strata of Y° . The first line says that I_0° is generated by the products of variables which, seen as a union of facets of $\nabla_{I_0}(I)$, geometrically contain the support of $T_{I_0}(I)$. The second line gives the unique irreducible decomposition of I_0° . The ideal of a maximal stratum $F^\circ \in T_{I_0}(I)$ is generated by all variables which, considered as facets of $\nabla_{I_0}(I)$, contain F° .

3.4.2. First order mirror degeneration. For a subcomplex C of complex of faces of Δ denote by $C^* \subset \Delta^*$ the co-complex of dual faces F^* with $F \in C$. We consider $\text{Strata}_\Delta(I_0)^* \subset \Delta^*$ as the co-complex of deformations of the mirror. The lattice points $\alpha \in \Xi = \text{supp}(\text{Strata}_\Delta(I_0)^*) \cap N$ of the support of this complex correspond via

$$0 \rightarrow N \xrightarrow{A^\circ} \mathbb{Z}^{\Sigma^\circ(1)} \xrightarrow{\deg} A_{n-1}(Y^\circ) \rightarrow 0$$

to degree 0 Cox Laurent monomials and represent degree 0 deformations $\varphi_\alpha \in \text{Hom}_{S^\circ}(I_0^\circ, S^\circ/I_0^\circ)_0$. Denote by $R^\circ \subset S^\circ$ the Picard-Cox ring of Y° .

Definition 12. The **first order tropical mirror** $\mathfrak{X}^\circ \subset Y^\circ \times \text{Spec } \mathbb{C}[s]/\langle s^2 \rangle$ of \mathfrak{X} is defined by the ideal

$$I^\circ = \langle m^\circ + s \cdot \sum_{\alpha \in \Xi} c_\alpha \cdot \varphi_\alpha(m^\circ) \mid m^\circ \in I_0^\circ \cap R^\circ \rangle \subset \mathbb{C}[s]/\langle s^2 \rangle \otimes S^\circ$$

with generic coefficients c_α .

Note, that it is sufficient to know a given family up to first order in the case of complete intersections (due to the Koszul complex resolution) and codimension 3 Gorenstein varieties (due to the theorem of Buchsbaum and Eisenbud, [14]).

4. APPLICATION TO GORENSTEIN COMPLETE INTERSECTIONS

4.1. Setup. We consider the setup of the mirror construction by Batyrev and Borisov [7] for complete intersections in Gorenstein toric Fano varieties. Let $Y = \mathbb{P}(\Delta)$ be a Gorenstein toric Fano variety of dimension n , represented by the reflexive polytope $\Delta \subset M_\mathbb{R}$, with normal fan $\Sigma \subset N_\mathbb{R}$ and Cox ring S . A disjoint union

$$\Sigma(1) = J_1 \cup \dots \cup J_c$$

is called a **nef partition** if all $E_j = \sum_{r \in J_j} D_r$ are Cartier, spanned by global sections. By $\sum_{j=1}^c E_j = \sum_{r \in \Sigma(1)} D_r = -K_Y$ general sections of $\mathcal{O}(E_1), \dots, \mathcal{O}(E_c)$ give a Calabi-Yau complete intersection $X \subset Y$.

4.2. Outline of the construction by Batyrev and Borisov. For the setup from Section 4.1 Batyrev and Borisov construct the mirror of X .

Proposition 13. [7] The polytopes $\Delta_j = \Delta_{E_j}$ of sections of E_j are lattice polytopes, and it holds

$$\Delta = \Delta_1 + \dots + \Delta_c$$

Define the lattice polytope ∇_j as the convex hull

$$\nabla_j = \text{convHull}(\{0\} \cup J_j)$$

and ∇ by

$$\nabla^* = \text{convHull}(\Delta_1 \cup \dots \cup \Delta_c)$$

Proposition 14. [7] It holds $\nabla = \nabla_1 + \dots + \nabla_c$.

In particular ∇ is a lattice polytope containing 0, hence:

Corollary 15. [7] The polytope ∇ is reflexive.

Let $\mathbb{P}(\nabla)$ be the Gorenstein toric Fano variety associated to ∇ . Then

$$\sum_{j=1}^c D_{\nabla_j} = -K_{\mathbb{P}(\nabla)}$$

is a nef partition, and X° given by general sections of $\mathcal{O}(D_{\nabla_1}), \dots, \mathcal{O}(D_{\nabla_c})$ is a Calabi-Yau complete intersection in $\mathbb{P}(\nabla)$.

Theorem 16. [6] *The Calabi-Yau complete intersections X and X° form a stringy topological mirror pair.*

A maximal projective subdivision $\bar{\Sigma}$ of $\Sigma = \text{NF}(\Delta)$ gives a maximal projective partial crepant desingularization

$$f : X(\bar{\Sigma}) \rightarrow \mathbb{P}(\Delta)$$

such that the T -divisors of the projective toric variety $X(\bar{\Sigma})$ correspond to the lattice points of the boundary of Δ^* . Then f induces a resolution $\bar{X} \rightarrow X$ of the complete intersection $X \subset \mathbb{P}(\Delta)$ such that \bar{X} is a complete intersection, has at most Gorenstein terminal abelian quotient singularities and $K_{\bar{X}} = \mathcal{O}_{\bar{X}}$. In particular, if $\dim(\bar{X}) \leq 3$, then \bar{X} is smooth.

4.3. Degenerations associated to complete intersections. The general nef complete intersection has a natural monomial degeneration using the Koszul complex resolution:

Lemma 17. [12] *Consider a nef partition $\Sigma(1) = J_1 \cup \dots \cup J_c$ as in the setup of Section 4.1,*

$$m_j = \prod_{v \in J_j} x_v \in S$$

and the reduced $\text{Pic}(Y)$ -generated monomial ideal

$$I_0 = \langle m_j \mid j = 1, \dots, c \rangle$$

Let $g_j \in S_{[E_j]}$ be general sections of $\mathcal{O}(E_j)$ (corresponding to a general linear combination of the lattice points of Δ_{E_j}) not involving monomials in I_0 . Then the $\text{Pic}(Y)$ -generated ideal

$$I = \langle f_j = t \cdot g_j + m_j \mid j = 1, \dots, c \rangle \subset \mathbb{C}[t] \otimes S$$

defines a flat family $\mathfrak{X} \subset Y \times \text{Spec}(\mathbb{C}[t])$ with fibers in Y and special fiber given by I_0 .

The deformations of I_0 are unobstructed and the base space is smooth. Let $v_1, \dots, v_p \in \text{Hom}(I_0, S/I_0)_0$ be a basis of the tangent space of the Hilbert scheme of X_0 . The degeneration \mathfrak{X} is general in the sense that if v is the tangent vector of \mathfrak{X} and $v = \sum_{i=1}^p \lambda_i v_i$, then we have $\lambda_i \neq 0 \forall i$.

4.4. Tropical construction of the Batyrev-Borisov mirror. We now apply the tropical mirror construction to the canonical degeneration \mathfrak{X} from Section 4.3 of a given nef complete intersection, and show that \mathfrak{X}° is the canonical degeneration associated to the Batyrev-Borisov mirror. For details see [12].

4.4.1. *Construction of the mirror polarization via Gröbner bases techniques.* As f_1, \dots, f_c form a reduced Gröbner basis with respect to any monomial ordering selecting I_0 as lead ideal, we get:

Lemma 18. *For the degeneration defined in Lemma 17 the special fiber Gröbner cone is*

$$C_{I_0}(I) = \{(w_t, w_x) \in \mathbb{R} \oplus N_{\mathbb{R}} \mid \text{trop}(g_j)(\varphi(w_x)) + w_t \geq \text{trop}(m_j)(\varphi(w_x)) \ \forall j\}$$

Hence the dual of the special fiber polytope

$$\nabla_{I_0}(I) = \{w_x \in N_{\mathbb{R}} \mid \text{trop}(g_j)(\varphi(w_x)) + 1 \geq \text{trop}(m_j)(\varphi(w_x)) \ \forall j\}$$

can be described as the convex hull of the lattice monomials $A^{-1}(\frac{m}{m_j})$ with the monomials m appearing in g_j for $j = 1, \dots, c$, hence:

Corollary 19. With the notation from Section 4.2 and the degeneration defined in Lemma 17

$$\nabla_{I_0}(I)^* = \text{convHull}(\Delta_1 \cup \dots \cup \Delta_c) = \nabla^*$$

So the mirror toric Fano variety Y° as defined in Section 3.3 coincides with $\mathbb{P}(\nabla)$ as defined by Batyrev and Borisov.

4.4.2. *Construction of the mirror degeneration via tropical geometry.* We now describe the tropical subcomplex $T_{I_0}(I) \subset \partial \nabla_{I_0}(I)$. As before denote by $\text{Strata}_\Delta(I_0)$ the subcomplex of toric strata of the boundary complex $\partial \Delta$ of Δ .

Theorem 20. *For the degeneration defined in Lemma 17 the map*

$$\begin{array}{ccccc} \nabla_{I_0}(I) & & \nabla_{I_0}(I)^* & & \Delta \\ \cup & & \cup & & \cup \\ T_{I_0}(I) & \rightarrow & T_{I_0}(I)^* & \rightarrow & \text{Strata}_\Delta(I_0) \\ F & \mapsto & F^* & \mapsto & \sum_{i=1}^c F^* \cap \Delta_i \end{array}$$

is an inclusion reversing bijection.

Note, that this in particular shows that the complex $T_{I_0}(I)$ is dual to the sphere $\text{Strata}_\Delta(I_0)$.

Proposition 21. Let $\Sigma^\circ(1) = J_1^\circ \cup \dots \cup J_c^\circ$ be the nef partition corresponding to the Batyrev-Borisov mirror and I_0° the associated monomial ideal as defined in Section 4.3. Then

$$\text{Strata}_\nabla(I_0^\circ) = T_{I_0}(I)$$

From Theorem 20 and Proposition 21 we obtain that the lattice points of the support of $\text{Strata}_\Delta(I_0)^* \subset \partial \Delta^*$ correspond to the first order deformations of I_0° . As X_0° is again a complete intersection, by the Koszul complex the first order tropical mirror family is a global flat family:

Theorem 22. *The tropical mirror degeneration of \mathfrak{X} (as introduced in Section 3.4.2) defines a flat family $\mathfrak{X}^\circ \subset Y^\circ \times \text{Spec } \mathbb{C}[s]$ and this coincides with the degeneration associated to the nef partition $\Sigma^\circ(1) = J_1^\circ \cup \dots \cup J_c^\circ$ (by Lemma 17).*

Of course, for a complete intersection it is sufficient to compute the mirror special fiber X_0° (as this again is a complete intersection). However the deformation data is necessary to describe the mirror degeneration in the case of non-complete intersections like, e.g., Pfaffian varieties.

Note also, that we have reproduced the Batyrev-Borisov mirror without the knowledge that it was a complete intersection, and without any non-trivial use of convex geometry (i.e., aside from convex hulls), as the tropical mirror construction directly obtains the relevant data $T_{I_0}(I)^* \subset \partial\nabla^*$ (see also the Algorithm formulated in Section 6).

5. EXAMPLE AND IMPLEMENTATION

We formulate the construction given in Section 4 in the form of an algorithm as implemented by the author in the MACAULAY2 [18] package SRDEFORMATIONS [13] (though the construction is not limited to complete intersections, but more complicated in general). In the complete intersection case we just have to specify the special fiber ideal I_0 and can obtain from that \mathfrak{X}° (and \mathfrak{X}).

Algorithm 1 Tropical mirror family

Input: Monomial ideal I_0 corresponding to a nef complete intersection Calabi-Yau variety in a Gorenstein toric Fano variety Y .

Output: The tropical mirror family $\mathfrak{X}^\circ \subset Y^\circ \times \text{Spec } \mathbb{C}[s]$

- 1: Compute a torus invariant basis of $\text{Hom}(I_0, S/I_0)_0$.
- 2: Compute the convex hull ∇^* of the lattice monomials corresponding to this basis.
- 3: Find the co-complex $T_{I_0}(I)^*$ of tropical faces of ∇^* , i.e., those faces F of ∇^* such that the ideal

$$\phi_F(I_0) = \langle m_0 + t \cdot \sum_{\alpha \in F \cap M} c_\alpha \cdot \phi_\alpha(m_0) \mid m_0 \in I_0 \rangle \subset \mathbb{C}[t] \otimes S$$

with generic coefficients c_α does not contain a monomial.

- 4: Via Equation 3.1 obtain the ideal I_0° associated to $T_{I_0}(I) \subset \partial\nabla$.
 - 5: **return** the first order tropical mirror family \mathfrak{X}° defined in Section 3.4.2.
-

Example 23. We treat the $K3$ surface given as the complete intersection of a quadric and a cubic in \mathbb{P}^4 using the MACAULAY2 package:

```
i1: R = QQ[x_0..x_4];
i2: I0 = ideal(x_0*x_1, x_2*x_3*x_4);
```

The Stanley-Reisner complex of I_0 :

```
i3: C = idealToComplex I0;
o3: 2:  x x x    x x x    x x x    x x x    x x x    x x x
      0 2 3    1 2 3    0 2 4    1 2 4    0 3 4    1 3 4
      complex of dim 2 embedded in dim 4 (printing facets)
      equidimensional, simplicial, F-vector {1,5,9,6,0,0}
```

Computing ∇^* as the convex hull of the deformations:

```
i4: NablaDual=PT1 C;
o4: 4:  y y y y y y y y y y
      0 1 2 3 4 5 6 7 8 9
```

```

    complex of dim 4 embedded in dim 4 (printing facets)
    equidimensional, non-simplicial, F-vector {1,10,24,25,11,1}
Compute  $T_{I_0}(I)^*$  as a subcomplex of the boundary of  $\nabla^*$ :
i5: TI0Dual = tropDef(C,NablaDual)
o5: 1:  y y      y y      y y      y y      y y
        0 4      8 9      3 7      2 6      1 5
    co-complex of dim 1 embedded in dim 4 (printing facets)
    equidimensional, non-simplicial, F-vector {0,0,5,9,6,1}
Dualize to obtain  $T_{I_0}(I)$  as a subcomplex of the boundary of  $\nabla$ :
i6: TI0 = dualize TI0Dual
o6: 2:  v v v      v v v v      v v v v      v v v v      v v v
        2 4 7      2 4 8 9      2 5 7 9      4 5 7 8      5 8 9
    complex of dim 2 embedded in dim 4 (printing facets)
    equidimensional, non-simplicial, F-vector {1,6,9,5,0,0}
The coordinates of the vertices of  $\nabla_{I_0}(I) = \nabla$ :
i7: transpose TI0.grading
o7:
      
$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & -1 & 0 & 1 & -1 & -1 & -1 \\ 0 & 1 & 1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & -1 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 & -1 \end{pmatrix}$$

i8: fvector C
o8: {1, 5, 9, 6, 0, 0}
i9: fvector B
o9: {1, 6, 9, 5, 0, 0}
We observe that  $T_{I_0}(I)$  and  $\text{Strata}_\Delta(I_0)$  have mirror dual  $F$ -vectors.

```

The code computing this example and others can be found in the documentation of the package SRDEFORMATIONS [13].

6. REMARKS AND FURTHER APPLICATIONS

The tropical mirror construction can also be applied for degenerations \mathfrak{X} of a non-generic complete intersection X to X_0 (defined by I_0) as long as $\nabla_{I_0}(I)$ will still be a polytope, e.g., to handle subfamilies with prescribed singularities.

The construction is also applicable, e.g., to non-complete intersection Gorenstein Calabi-Yau varieties of codimension 3, indeed, handling non-complete intersection cases is the main aim of the construction. In particular, as will be treated in a separate paper, it can be used to reproduce a known mirror construction for a Pfaffian Calabi-Yau by Rødland [29] and yields new examples of non-complete intersection mirror pairs, see, e.g., [12, Sec. 10].

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